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FUNDAMENTAL LIMITATIONS ON IMAGE RESTORATION

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The objective of this program is the analysis of the fundamental limitations placed upon image restoration by the presence of noise and unavoidable atmospheric and optical system degradations. Our approach is to treat individually and in concert the several factors which limit image formation (direct problem) and further limit image restoration (inverse problem), to understand their interrelation and order of importance. Within this overall framework, we have treated three problem areas comprising the chapters of this

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FUNDAMENTAL LIMITATIONS ON IMAGE RESTORATION

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SUMMARY

The objective of this program is the analysis of the fundamental limitations placed upon image restoration by the presence of noise and unavoidable atmospheric and optical system degradations. Our approach is to treat individually and in concert the several factors which limit image formation (direct problem) and further limit image restoration (inverse problem), to understand their interrelation and order of importance. Within this overall framework, we have treated three problem areas comprising the chapters of this report: sums of random amplitudes individually governed by a lognormal probability density function, realizability constraints on wavefront covariance, and (partial) moment behavior of the transfer function random process, including temporal effects and those due to random amplitude errors acting in conjunction with random phase errors.

The lognormal probability density permeates much of the current literature on optical propagation through the turbulent atmosphere and has been the source of a good deal of controversy in various contexts of the general problem. Sums of lognormal random variables arise through aperture averaging and other related processes. Because the lognormal density assumes central importance in many atmospheric propagation problems, certain aspects of the behavior of (sums of) lognormally distributed variables are considered in Chapter I. Particular results include the derivation of an expression for the characteristic function of the lognormal probability density, which is then used to calculate densities for sums of lognormally distributed random variables; a simplified proof that the lognormal probability density is *not* determined by its moments even though they exist; an investigation of the observed "permanence" of the lognormal density in terms of

its asymptotic behavior, leading to a soundly based explanation of the phenomenon.

Realizability conditions on the covariance of the wavefront aberration function are briefly stated in Chapter II; the class of admissible covariance functions is restricted to those representing (isotropic) random surfaces. The commonly used Kolomogorov five-thirds spectrum is shown to violate the stated realizability conditions, but no severe practical consequence normally arises since the sampling interval is finite.

Investigations of the statistics of the transfer function random process, predominantly restricted in our prior work to errors of phase, have been extended to include the statistical dependence of the transfer function process on time (and concurrently on integration time), as well as on jointly present random amplitude and phase errors. Results show that the expected value of the time dependent transfer function *does not* depend on integration time, but that the general higher moment behavior *is* collection time dependent. Random amplitude effects are considered within the usual lognormal and normal assumptions regarding the respective amplitude and phase probability densities, but without the added restriction of independence between the amplitude and phase random processes. In terms of the expected transfer function, a product of four terms arises: one standard deterministic part, one part due to phase errors acting alone, one part due solely to amplitude perturbations, and a cross term which is of unit modulus and depends for its existence on the odd part of the cross correlation between amplitude and phase.

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ABSTRACT

This report is divided into three chapters which treat three problem areas pertinent to optical propagation through the turbulent atmosphere. In the first chapter, several aspects of the lognormal probability density, which permeates much of the current literature and which has been the source of a good deal of controversy in various relevant contexts, are treated. Particular developments include the derivation and use of an expression for the characteristic function of the lognormal probability density, a simplified proof that the lognormal probability density function is *not* determined by its moments, and an investigation of the observed permanence of the lognormal distribution via an adaptation of the Cramer asymptotic formula for sums of independent random variables having nonzero means. The second chapter of the report concerns realizability constraints on the covariance of the wavefront aberration function, which are briefly stated; it is shown that the class of admissible covariance functions is restricted to those that represent a (isotropic) random surface. The commonly used Kolomogorov five-thirds spectrum is seen to violate the stated realizability conditions, but the practical significance of the violation is not excessive due to the limited sampling interval. In chapter III, the effects of integration time-dependence and random amplitude on the optical transfer function are treated in terms of the moments of the transfer function random process. It is shown that the expected value of the time-dependent transfer function is not a function of the integration time interval, but that general higher moments (explicitly, second order averaged products) *do* depend on collection time. Random amplitude effects are considered for the case of lognormal amplitude and Gaussian phase errors, without assuming independence tween the amplitude and phase perturbations. In terms of the expected transfer function, a product of four terms arises: one deterministic part, one part due to phase errors acting alone, a similar part due only to amplitude errors, and a cross term which is of unit modulus and which depends for its existence on the odd part of the cross correlation between log amplitude and phase.

CHAPTER I

SUMS OF INDEPENDENT LOGNORMALLY DISTRIBUTED RANDOM VARIABLES

1. INTRODUCTION

The lognormal probability distribution permeates much of the current literature[1] on optical propagation through the turbulent atmosphere and has been the source of a good deal of controversy in various contexts of the general problem. The purpose of the present paper is to report on three aspects of the lognormal probability density function which are pertinent to the atmospheric propagation problem.

The three topics are:

1. The development of an expression for the characteristic function of the lognormal probability density function and the use of the characteristic function to calculate sums of lognormally distributed random variables.

2. Simplified proof, via the Carleman criterion, that the lognormal probability density function is *not* determined by its moments even though they exist. A discussion of the consequence of this result as regards theoretical approaches to optical wave propagation in random atmospheres.

3. The "permanence" of the lognormal is investigated via an adaption of the Cramer asymptotic formula for sums of independent random variables having nonzero means.

2. CHARACTERISTIC FUNCTION OF LOGNORMAL PDF

Consider the sum

$$x = \sum_{k=1}^{N} x_k \quad , \tag{1}$$

where the x_k are independent random variables all having the identical lognormal probability density function (PDF)

$$f(x_k) = \frac{1}{x_k \sigma \sqrt{\pi}} e^{-(\log x_k - \xi)^2/2\sigma^2} \quad , \quad x_k \geq 0 \tag{2}$$

where

$$\xi = E(\log x_k) \equiv \langle \log x_k \rangle$$

$$\sigma^2 \equiv \mathrm{var}(\log x_k) = E[(\log x_k - \xi)^2] \quad . \tag{3}$$

As a matter of typographic convenience, we will use $E(\cdot)$ and $\langle \cdot \rangle$ interchangeably. It is somewhat more convenient to rewrite Eq. 2 in the form

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-[\log(x/x_0)]^2/2\sigma^2} \quad , \tag{4}$$

where

$$x_0 = e^{\xi} \quad . \tag{5}$$

The probability distribution of the sum x, call it $f_N(x)$, is most easily obtained as the Fourier transform of the *Nth* power of the characteristic function. The characteristic function of $f(x_k)$ is

$$E\left[e^{itx_k}\right] \equiv \phi_{x_k}(t) = \int_0^\infty f(x_k) e^{itx_k} dx_k \quad , \tag{6}$$

so that

$$f_N(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \left[\phi_{x_k}(t)\right]^N dt \quad . \tag{7}$$

The usefulness of this approach is flawed by the fact that the characteristic function of the lognormal PDF is unknown (at least, it has never appeared in the literature). Therefore we have carried out an analysis which yields the characteristic function, an outline of the analysis is presented in the next few paragraphs.

Upon setting $x_k/x_0 = y$ in Eq. 2, we can write Eq. 6 in the form

$$\phi_{x_k}(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} e^{ix_0 te^y} dy \quad . \tag{8}$$

Note that ξ enters only through the product $\tau = x_0 t$. This means that for a given σ, $\phi_y(\tau)$ need only be evaluated once since different values of ξ or x_0 merely rescale the characteristic function.

The main contribution to the integral comes from the integrand evaluated in the vicinity of $y = 0$. This suggests that we expand

$$e^{i\tau e^y} = e^{i\tau} e^{i\tau y} [e^{i\tau y^2/2} e^{i\tau y^3/6} \ldots]$$

$$= e^{i\tau} e^{i\tau y} \sum_{n=0}^{\infty} b_n(i\tau) y^n \quad , \tag{9}$$

and integrate termwise. However, the explicit derivation of the b-coefficients rapidly becomes unmanageable.

A more systematic approach to obtaining the equivalent of the b-coefficients is possible provided we rewrite Eq. 6 in the form

$$\phi_{x_k}(t) = \frac{e^{i\tau}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} e^{i\tau(e^y-y-1)} dy \quad . \tag{10}$$

Now expand

$$\exp[i\tau(e^y-y-1)] = \sum_{n=0}^{\infty} \frac{1}{n!} a_n(i\tau) y^n \quad , \tag{11}$$

where

$$a_n(i\tau) = \frac{d^n}{dy^n} \exp[e^{i\tau}(e^y-y-1)]_{y=0} \quad , \tag{12}$$

These derivatives, although extremely tedious if attempted by hand, are easily obtained on a computer using symbol manipulation programs. We employed the symbol manipulation program available at MIT (Project MAC) and the results for the first twelve a_n are listed in Table 1 where $s \equiv i\tau$.

When the series in Eq. 11 is substituted in Eq. 10, we encounter the integrals:

$$\int_0^\infty y^{2m+1}\, e^{-y^2/2\sigma^2} \sin\tau y\, dy = (-1)^m \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sigma^{2m+2}\, e^{-\tau^2\sigma^2/2}\, h_{2m+1}(\sigma\tau) \,, \tag{13}$$

$$\int_0^\infty y^{2m}\, e^{-y^2/2\sigma^2} \cos\tau y\, dy = (-1)^m \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sigma^{2m+1}\, e^{-\tau^2\sigma^2/2}\, h_{2m}(\sigma\tau) \,, \tag{14}$$

where

$$h_n(x) \equiv He_n(x) = (-1)^n\, e^{x^2/2} \frac{d^n}{dx^n}\, e^{-x^2/2} \quad . \tag{15}$$

Note that these Hermite polynomials are different than those customarily employed in theoretical physics. The first few are:

$$h_0(x) = 1 \quad , \qquad h_3(x) = x^3-3x$$

$$h_1(x) = x \quad , \qquad h_4(x) = x^4-6x^2+3$$

$$h_2(x) = x^2-1 \quad , \qquad h_5(x) = x^5-10x^3+15x \qquad (16)$$

The final expression for the characteristic function is

$$\phi_{x_k}(\tau) = e^{i\tau} e^{-\tau^2\sigma^2/2} \sum_{n=0}^{\infty} \frac{(i\sigma)^n}{n!} a_n(i\tau) h_n(\sigma\tau) \quad . \qquad (17)$$

It is a complex-valued function of τ since $f(x_k)$ is nonzero only over the semi-infinite interval $(0,\infty)$. The series converges very slowly (if at all) for large σ, but practically the only values of σ encountered in random wave propagation are for $\sigma < 1$. Although it is not the purpose of this paper to present elaborate numerical calculations, we ran a series of tests for $\sigma = .25$ varying the number of terms M in the series and found that $M = 9$ was sufficient to guarantee convergence. A plot of $\phi_{x_k}(t)$ is shown in Fig. 1; note the oscillatory behavior.

Given this information, the numerical evaluation of $f_N(x)$ via Eq. 7 is difficult because the integrand is oscillatory. Furthermore since $f_N(x)$ will be highly skewed to the right of its maximum, it is imperative that a quadrative formula be employed which yields *uniform* accuracy for large values of x. We refer the interested reader to Appendix A for details of this aspect of the problem.

On the principal uses of the characteristic function is as a moment generating function since

$$m_k \equiv \langle x^k \rangle = (-1)^k \left. \frac{\partial^k \phi_x}{\partial t^k} \right|_{t=0} \tag{18}$$

Unfortunately Eq. 17 is too complicated for this purpose, but it is of no great importance since the moments of the lognormal PDF are well known. The moments about the origin are

$$m_k = e^{k\xi} e^{k^2\sigma^2/2} \quad , \tag{19}$$

and increase very rapidly with k. The central moments are

$$\mu_k = E[(x-m_1)^k] \quad , \tag{20}$$

and Wartmann has shown that

$$\mu_k = \frac{(\mu_2)^{k/2}}{(\omega-1)^{k/2}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \omega^{(k-j)(k-j-1)/2} \quad , \tag{21}$$

where $\omega \equiv \exp \sigma^2$ and

$$\mu_2 = \omega(\omega-1)e^{2\xi} \quad . \tag{22}$$

We will utilize these formulae shortly.

The coefficient of skewness γ_1 and the coefficient of excess γ_2 are:

$$\gamma_1 \equiv \frac{\mu_3}{\mu_2^{3/2}} = \frac{\omega^{3/2}(\omega-1)^2(\omega+2)e^{3\xi}}{\omega^{3/2}(\omega-1)^{3/2}e^{3\xi}} = (\omega-1)^{\frac{1}{2}}\,(\omega+2) \quad , \tag{23}$$

$$\gamma_2 \equiv \frac{\mu_4-3}{\mu_2^2} = \frac{\omega^2(\omega-1)^2(\omega^4+2\omega^3+3\omega^2-3)e^{4\xi}}{\omega^2(\omega-1)^2e^{4\xi}}$$

$$= \omega^4 + 2\omega^3 + 3\omega^2 - 6 \quad . \tag{24}$$

Neither γ_1 or γ_2 depends on ξ. Furthermore $\gamma_1 > 0$, thereby indicating a long tail to the right of the maximum of the lognormal PDF. Some numerical values of γ_1 and γ_2 are given in Table 2.

3. NONUNIQUENESS OF MOMENT PROBLEM FOR LOGNORMAL PDF

The determination of a PDF via its moments (provided they exist) is a time established method. In other words does a set of moments determine a PDF? The known answer is that at least one PDF is determined, but there may be others (i.e., the solution may not be unique). There is a tendency among practical users of the moment method to assume that if the moments exist then they determine a PDF uniquely, such is not the case. *In point of fact the lognormal PDF cannot be uniquely defined by its moments*!

The Carleman criterion[2] for the uniqueness of the PDF defined over $(0,\infty)$ via a moment sequence is that

$$\sum_{k=1}^{\infty} (\mu_k)^{-1/k} = \infty \quad , \tag{25}$$

in other words the series of central moments must diverge. It is not difficult to show that μ_k increases rapidly enough for the series in Eq. 25 to *converge*, thereby violating the Carleman uniqueness criterion.

This indeterminate nature of the lognormal moment problem manifests itself in two important ways.

First, any theory predicting the probability density of the irradiance fluctuations of an optical wave which has as its primary goal the calculations of the moments must be checked to

see if the Carleman uniqueness criterion is valid. In particular any theory which predicts a lognormal probability density for the irradiance fluctuations *strictly on the basis of the irradiance fluctuation moments* must be suspect. This lends credence to the contention of Klyatskin[3] that there is no limiting form for the probability density in the region of strong fluctuations, although the criticisms of DeWolf[4] concerning the macroscales of turbulence are well taken.

Second, there is the very interesting and provocative paper by Mitchell[5] on the "permanence of the lognormal distribution." Mitchell's thesis is that the sum of independent lognormally distributed random variables is approximately lognormal in apparent contradiction to the central limit theorem which predicts a Gaussian distribution. He employs an analysis which depends upon the moments of the lognormal PDF. Since the moments of the sum are the sum of the moments, the Carleman criterion is violated. Actually there is no contradiction and one can easily derive the main points of Mitchell's paper by standard arguments as we will demonstrate in the next section.

As an historical note we mention that Heyde[7] was evidently the first person to publish the fact that the lognormal is not determined by its moments. His proof is rather involved and does not employ Eq. 25.

4. QUASIPERMANENCE OF THE LOGNORMAL PDF

Let us now consider the problem of determining the correction terms to the central limit theorem so that we can establish the rate of convergence to the Gaussian PDF. Fortunately, Cramer[7] has made an especially detailed study of this problem which seems to have escaped the notice of many workers in applied probability theory, yet is of direct importance insofar as our problem is concerned. Cramer's analysis is fairly general and we will specialize his approach to the case where all the random variables possess exactly the same PDF. In the Cramer approach, the PDF of the sum of the independent random variables is given as the product of the limiting Gaussian PDF and an *asymptotic* expansion in powers of $N^{-\frac{1}{2}}$.

Let us return to Eq. 1 and note that since the x_k are independent random variables, then the mean and variance of the sum x is

$$\langle x \rangle = N\xi \ , \qquad \text{var } x = N\sigma^2 \ . \tag{26}$$

It is convenient to rescale the random variable x by defining new random variables z_k and z

$$z_k = \frac{x_k - \xi}{\sigma} \ , \qquad z = \frac{x - N\xi}{\sigma\sqrt{N}} \tag{27}$$

so that

$$\langle z\rangle = 0 \ , \qquad \mathrm{var}\ z = 1 \ . \tag{28}$$

The characteristic function of z is

$$E[e^{itz}] \equiv \phi_z(t) = [\phi_{z_k}(t)]^N \tag{29}$$

but

$$\phi_{z_k}(t) = \int_{-\infty}^{\infty} e^{iz_k t}\ f(z_k)dz_k$$

$$= \sum_{n=0}^{\infty} \frac{(it)^n\langle(x_k-\xi)^n\rangle}{\sigma^n n!\, N^{n/2}} \tag{30}$$

hence,

$$\phi_z(t) = \left[1 - \frac{t^2}{2N} - \frac{it^3\mu_3}{6\sigma^3 N^{3/2}} + \frac{t^4\mu_4}{24\sigma^4 N^2} + 0(N^{-5/2})\right]^N \ . \tag{31}$$

Taking the logarithm of both sides yields

$$\log\phi_z(t) = -\frac{t^2}{2} - \frac{it^3\mu_3}{6\sigma^3 N^{1/2}} + \left(\frac{\mu_4-3}{24\sigma^4 N}\right)t^4 + 0(N^{-5/2}) \tag{32}$$

where we have employed

$$\log(1 + s) \simeq s - \frac{s^2}{2} + \frac{s^3}{3} \ , \qquad (s < 1) \ . \tag{33}$$

Consequently,

$$\phi_z(t) = e^{-t^2/2}\left[1 - \frac{i\gamma_1 t^3}{6N^{\frac{1}{2}}} + \frac{\gamma_2 t^4}{24N} - \frac{\gamma_1^2 t^4}{72N} + O(N^{-\frac{3}{2}})\right] . \qquad (34)$$

Note that

$$\phi_z(0) = 1 \ , \qquad \phi_z(t) = \phi_z^*(-t) \qquad (35)$$

so it is truly a characteristic function.

The Fourier transform of $\phi_z(t)$ is $f_N(z)$

$$f_N(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_z(t)\ e^{-izt} dt \ . \qquad (36)$$

Upon substituting Eq. 34 into Eq. 36 we can show that

$$f_N(z) \sim \frac{1}{\sqrt{2\pi}}\ e^{-z^2/2}\left[1 - \frac{\gamma_1}{6N^{\frac{1}{2}}}\ h_3(z) + \frac{\gamma_2}{24N}\ h_4(z) + \frac{\gamma_1^2}{72N}\ h_6(y) + O(N^{-\frac{3}{2}})\right] \qquad (37)$$

where the $h_k(z)$ are the hermite polynomials defined in Eq. 15 and γ_1, γ_2 are the coefficients of skewness and excess respectively, as defined in Eqs. 23, 24. An equivalent expression is

$$f_N(z) \sim \phi(z) - \frac{\gamma_1}{6N^{\frac{1}{2}}}\ \phi^{(3)}(z) + \frac{\gamma_2}{24N}\ \phi^{(4)}(z) + \frac{\gamma_1^2}{72N}\ \phi^{(6)}(z) + O(N^{-\frac{3}{2}}) \qquad (38)$$

where

$$\phi(z) \equiv \frac{1}{\sqrt{2\pi}}\ e^{-z^2/2} \qquad (39)$$

and $\phi^{(n)}(z)$ denotes the *nth* derivative with respect to z. This formula is very useful because $\phi(z)$ and its derivatives are extensive tabulated. Equations 37 and 38 are special cases of Cramer's general formula.

The leading term of Eq. 37 is the dominant term as N approaches infinity and is the Gaussian PDF of the central limit theorem. However, for large finite N, the other terms exert influence. The first correction term proportional to $N^{-\frac{1}{2}}$ contains the skewness coefficient γ_1 and is the dominant correction term. Most investigations, such as for sums of independent random sine waves, involve PDF's that are symmetric so that $\gamma_1 \equiv 0$. In that case, the degree of approximation is governed by the coefficient of excess γ_2 which decays as N^{-1}. Thus sums of independent random variables having nonzero mean converge very slowly (i.e., as $N^{-\frac{1}{2}}$) to the Gaussian PDF. This is the situation encountered when the random variables are lognormally distributed. Furthermore, when $\gamma_1 \neq 0$, then $f_N(z)$ is not symmetric since $h_3(z) = -h_3(-z)$.

In order to calculate the skewness coefficient $\gamma_1(z)$ for $f_N(z)$, we can easily show that the third moment of z is

$$\langle z^3 \rangle = -\frac{\gamma_1}{6N^{\frac{1}{2}}} \int_{-\infty}^{\infty} (z^6 - 3z^4) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz + O(N^{-\frac{3}{2}})$$

$$= -\frac{\gamma_1}{N^{\frac{1}{2}}} + O(N^{-\frac{3}{2}}) \ . \tag{40}$$

Consequently, the skewness coefficient for $f_N(z)$ is

$$\gamma_1(z) = -\frac{(\omega-1)^{\frac{1}{2}}(\omega+2)}{N^{\frac{1}{2}}} + O(N^{-\frac{3}{2}}) \tag{41}$$

where we have employed Eq. 23. The skewness coefficient for the sum decreases only as $N^{-\frac{1}{2}}$ and the tail to the recultant PDF is very noticeable for large N. The reason for the negative sign lies in the definition of the normalized variable z since z can only be positive when $x > N\xi$.

Thus $f_N(z)$ is a PDF with a long tail and it *looks* as if it were lognormally distributed. It is in this sense that the lognormal PDF is "quasipermanent" since the decay to the limiting Gaussian PDF is so slow.

APPENDIX A

The evaluation of Eq. 7 is approached by noting that $\phi(t)$ essentially vanishes for very large t; in Fig. 1 the maximum value of t for which $\phi(t)$ is nonzero is $t = 20$. Thus we can write

$$f(x) = \frac{1}{2\pi} \int_{-T}^{T} \phi(t)\, e^{-ixt} dt \qquad (A.1)$$

where T is to be determined according to the specific situation.

For large values of $|x|$, the graph of the integrand consists of positive and negative areas of nearly equal size. The addition of these two areas results in substantial cancellations with loss of accuracy. In addition, a very fine mesh spacing of quadrature points is necessary; since for large $|x|$, the integrand oscillates very rapidly. Unfortunately for us, it is precisely in this region of large x that reasonably accurate values of the integral are required. The method developed by the author is based upon the fact that f(x) is basically a (finite) Fourier integral. Now in the advanced mathematical literature on numerical analysis, there is a theorem which roughly states that the trapezoidal rule of numerical integration is *in the limit of very small mesh size* the way to

evaluate Fourier integrals. Unfortunately the mesh size must be vanishingly small which means that the number of quadrature points is extremely large (tending to infinity). This fact mitigates against the literal use of the trapezoidal rule, although the Fast Fourier Transform (FFT) method is essentially a naive version. Nevertheless, some modification of the trapezoidal rule to finite mesh sizes consistent with our accuracy requirements is possible. Furthermore we really need algorithm which retrains uniform accuracy when $|x|$ is large.

The ordinary trapezoidal rule when applied to Eq.7 yields

$$f(x) = \frac{1}{2\pi M} \sum_{m=-M}^{M} H_m \phi(m\delta t) e^{-imx\delta t} \tag{A.2}$$

where $M\delta t = T$ and $(2M + 1)$ = number of quadrature points. The weight factors are

$$\begin{aligned} H_m &= 1 \quad , \qquad m \neq \pm M \\ &= \frac{1}{2} \quad , \qquad m = \pm M \end{aligned} \tag{A.3}$$

This well known algorithm is based upon the hypothesis that the *entire* integrand varies linearly over the mesh interval δt so that it cannot be usefully employed unless $|x\delta t| << |$. For when $|x\delta t|$ does not satisfy this stringent requirement, the integrand varies nonlinearly over δt.

However let us still retain the trapezoidal rule but modify it by requiring that *only* $\phi(t)$ vary linearly over the interval δt. In this manner we can separate out the strong influence of large $|x|$ on the oscillations of the entire integrand. In a separate paper (to be published) the author has shown that under this restriction the weight factors are no longer constants but depend on x:

$$H_m = \left[\frac{\sin\frac{1}{2}x\delta t}{\frac{1}{2}x\delta t}\right]^2, \qquad m \neq \pm M$$

$$= \left[\frac{1-\cos x\delta t}{(x\delta t)^2}\right] + i\left[\frac{x\delta t - \sin x\delta t}{(x\delta t)^2}\right], \qquad m = +M$$

$$= \left[\frac{1-\cos x\delta t}{(x\delta t)^2}\right] - i\left[\frac{x\delta t - \sin x\delta t}{(x\delta t)^2}\right], \qquad m = -M \tag{A.4}$$

Note that these weight coefficients tend to those of the regular trapezoidal rule as $(x\delta t)\to 0$. The fact that only $\phi(t)$ has to vary linearly over δt means that we can take the mesh size to be relatively coarse in many cases of practical interest.

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TABLE 1. FIRST TWELVE a_n COEFFICIENTS

$$a_0(s) = 1$$

$$a_1(s) = 0$$

$$a_2(s) = s$$

$$a_3(s) = s$$

$$a_4(s) = 3s^2 + s$$

$$a_5(s) = 10s^2 + s$$

$$a_6(s) = 15s^3 + 25s^2 + s$$

$$a_7(s) = 105s^3 + 56s^2 + s$$

$$a_8(s) = 105s^4 + 490s^3 + 119s^2 + s$$

$$a_9(s) = 1260s^4 + 1918s^3 + 246s^2 + s$$

$$a_{10}(s) = 945s^5 + 9450s^4 + 6825s^3 + 501s^2 + s$$

$$a_{11}(s) = 17325s^5 + 56980s^4 + 22935s^3 + 1012s^2 + s$$

TABLE 2. NUMERICAL VALUES OF γ_1 AND γ_2 AS A FUNCTION OF σ.

σ	γ_1	γ_2
.1	.30	.16
.2	.61	.68
.3	.95	1.64
.4	1.32	3.26
.5	1.75	5.90
.6	2.26	10.3
.7	2.89	17.8
.8	3.69	31.4
.9	4.75	57.4
1.0	6.18	110.9

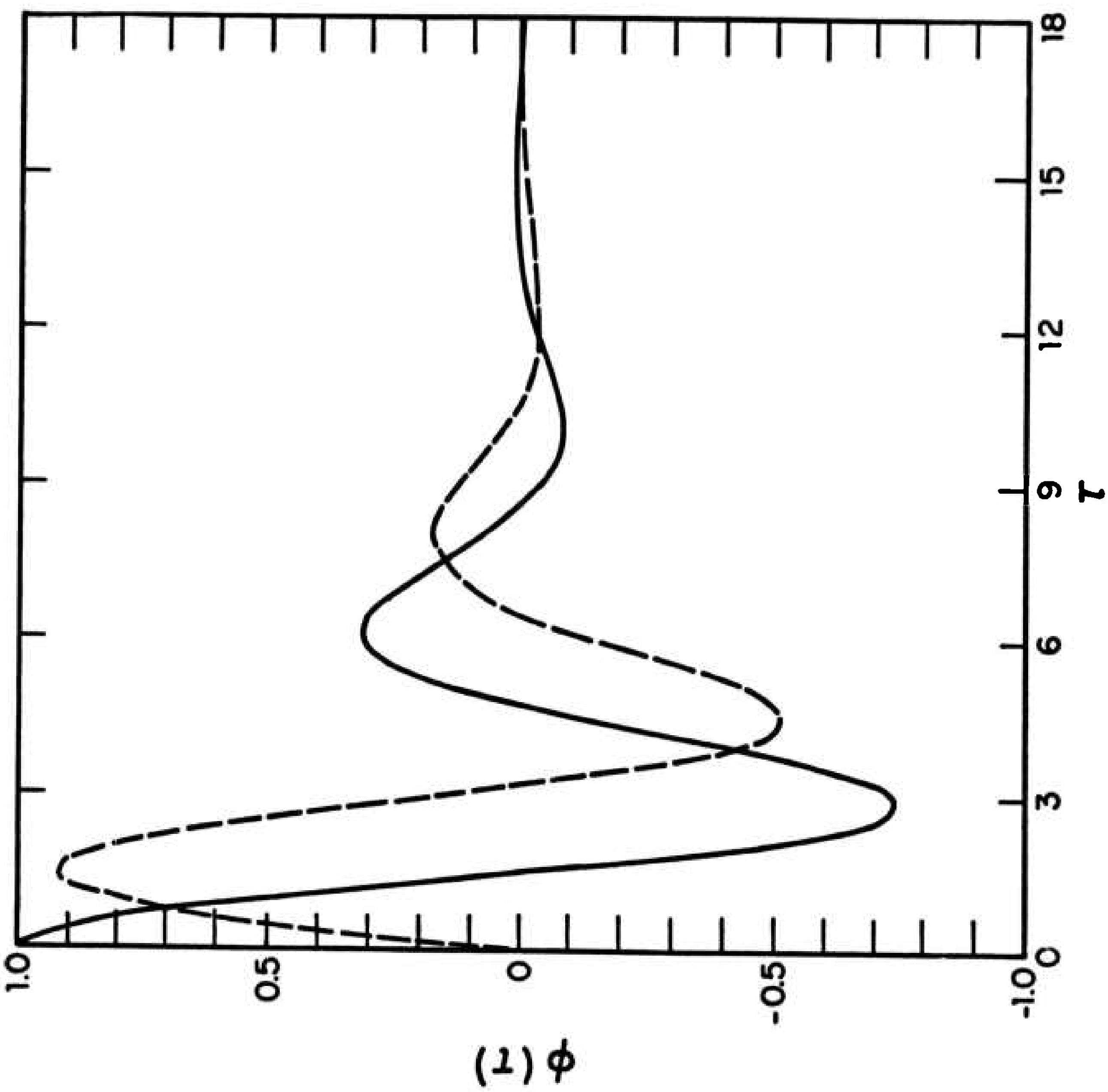

Figure 1. Characteristic function $\phi_x(\tau)$ of lognormal probability density function with $\sigma^2 = 0.25$: —— real part of $\phi_x(\tau)$, ---- imaginary part of $\phi_x(\tau)$.

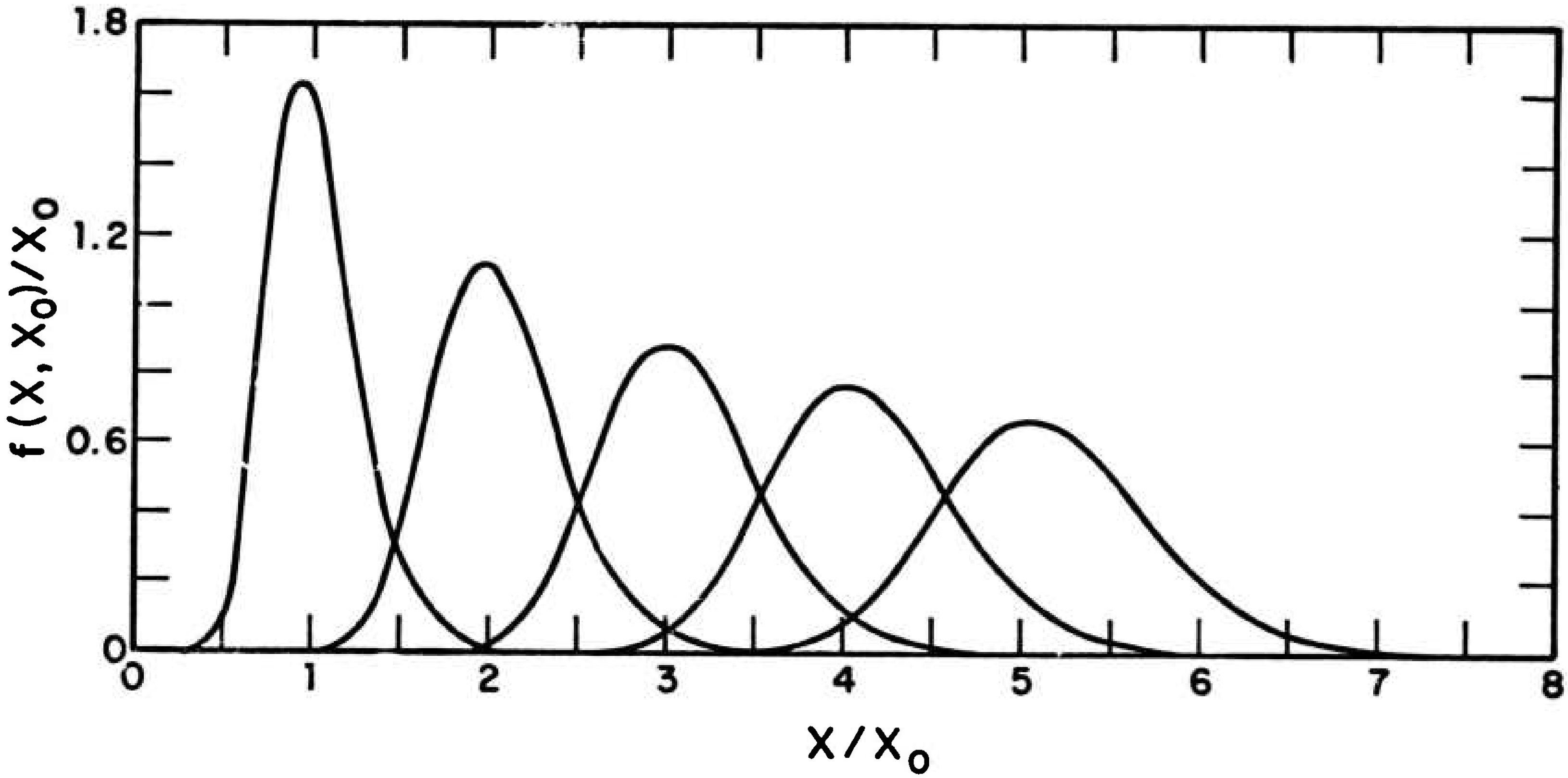

Figure 2. Probability density $f_N(x)$ of the sum of N independent lognormally distributed random variables with $\sigma^2 = 0.25$ and N = 1,2,3,4,5.

CHAPTER II

REALIZABILITY CONDITIONS ON THE COVARIANCE OF THE WAVEFRONT ABERRATION FUNCTION

The study of the influence of random wavefront aberration functions on the imaging characteristics of optical systems is a subject of current interest. The primary quantity that enters into the analysis is the covariance function of the random wavefront aberration function. The purpose of this note is to show that the class of admissable covariance functions is restricted to those that represent an isotropic random surface.

The complex diffracted amplitude $a(x,y)$ in the specified receiving plane is

$$a(x,y) = \iint\limits_{\text{aperture}} A_0(p,q)\, e^{ikW(p,q)}\, e^{i(xp+yq)}\, dpdq \tag{1}$$

where $A_0(p,q)$ is the amplitude distribution over the exit pupil and $W(p,q)$ is the wavefront aberration function (Hamilton's mixed characteristic). Both A_0 and W are real functions of p,q.

When the wavefront W is spatially random then it consists of two terms

$$W(p,q) = W_D(p,q) + W_s(p,q) \tag{2}$$

where W_D is the usual deterministic part due to the aberrations of the optical system and W_s is the stochastic part due to atmospheric turbulence. W_s is taken to be a spatially stationary (homogeneous), Gaussian random process having a zero mean and a specified covariance.

$$E[W_s(p,q)] = 0 \tag{3}$$

$$E\left|W_s\left(p + \frac{1}{2}\alpha_1, q + \frac{1}{2}\beta_1\right) W_s\left(p - \frac{1}{2}\alpha_2, q - \frac{1}{2}\beta_2\right)\right|$$

$$= \sigma^2 r(\alpha_1 - \alpha_2, \beta_1 - \beta_2) \ . \tag{4}$$

Only the difference of the independent variables enters because W_s is homogeneous. In the vast majority of cases, W_s is also isotropic so that the correlation function r becomes a function only of $\rho = [(p_1-p_2)^2 + (q_1-q_2)^2]^{\frac{1}{2}}$.

Since $W_s(p,q)$ is a surface (it is the surface of constant phase), then its slope and curvature must exist and these requirements place a restriction upon admissable forms of its correlation function. In the context of the mean square stochastic calculus, the requirements translate into conditions on the derivatives of the isotropic correlation function. Longuet-Higgins (1) has shown that isotropic, Gaussian random surfaces can be described by three invariants m_0, m_2, and m_4 which can be expressed in terms of the isotropic correlation function

$$m_{2n} = \left.\frac{d^{2n}r(\rho)}{d\rho^{2n}}\right|_{\rho = 0} , \qquad n = 0,1,2 \ . \tag{5}$$

Consequently, in order that the three parameters exist, the isotropic correlation function of W_s must be smooth at the origin

$\rho = 0$ in the sense that it and its second and fourth derivatives exist.

The Gaussian correlation function

$$r(\rho) = e^{-a\rho^2} , \qquad a > 0 \tag{6}$$

employed by Barakat and Blackman (2,3,4) obviously satisfies the differentiability conditions since

$$r''(0) = (2a)^2 ; \qquad r''''(0) = (2a)^4 . \tag{7}$$

Not so with the Kolmorogov five-thirds spectrum employed by Roddier and Roddier (5)

$$r(\rho) = 1- \rho^{5/3} . \tag{8}$$

Here

$$r''(\rho) = - \frac{10}{9} \rho^{-1/3}$$

$$r''''(\rho) = \frac{40}{81} \rho^{-7/3} \tag{9}$$

and both derivatives become infinite (and thus do not exist) at $\rho = 0$. There is an inherent *theoretical* contradiction in the Roddiers' analysis since their random wavefront does not allow slopes and curvatures to exist though they proceed to obtain numerical data. The reason why this does not amount to a contradiction in *practice* is that the sampling interval is finite; the

effect of a finite sampling interval being to filter out high frequency (i.e., short wavelength) components and thus alter the behavior of the covariance function at the origin.

An interesting variant of these conditions holds for the slit aperture, here

$$a(x) = \int_{-1}^{+1} A_0(p)\, e^{ikW(p)}\, e^{ixp}\, dp \; . \tag{10}$$

Now $W(p)$ being one-dimensional only the slope condition is required. This means that the permissible forms of the covariance function of $W(p)$ need only possess finite second derivatives at $\rho = 0$.

The question naturally arises as to what happens when $A_0(p,q)$ is also random. Now $A_0(p,q)$ is not a surface in the sense that $W(p,q)$ is and there are no additional differentiability requirements on its covariance function.

These differentiability conditions become much more complicated when $W(p,q)$ is nonisotropic and then involve nine functions not just three but they still only involve the equivalent of the fourth moment.

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CHAPTER III

STATISTICS OF THE TRANSFER FUNCTION: TEMPORAL AND RANDOM AMPLITUDE EFFECTS

1. INTRODUCTION

In prior work[1,2] we have investigated the behavior of the transfer function of an optical system subject to random wavefront errors, by developing explicit formulae for the moments of the transfer function random process, and by computing the low order moments in addition to simulating the process in a digital computer. Analysis was predominantly restricted to errors of phase, i.e., effects due to random amplitude errors were neglected. Also, statistical dependence of the transfer function process on time (and concurrently on integration time) was not part of the formulation. As the formation of images through intervening turbulence is a present and practical problem at hand, considerations of temporal effects and those of random amplitude errors should rightly be made. It is the inclusion of these considerations, temporal dependence and random amplitudes, within the framework of our prior formalism, to which this chapter is addressed.

2. TEMPORAL EFFECTS

Before explicitly treating the temporal behavior of the transfer function, we review the time-independent formulation. To aid clarity, only one spatial (reduced) coordinate is treated. Following Barakat and Blackman[2], and ignoring *here* random amplitude errors,

$$T(\alpha) = \frac{1}{2} \int_{-\left(1 - \frac{|\alpha|}{2}\right)}^{1 - \frac{|\alpha|}{2}} \exp\left\{ki\left[W\left(p + \frac{\alpha}{2}\right) - W\left(p - \frac{\alpha}{2}\right)\right]\right\}dp \tag{1}$$

with $|\alpha| < 2$. Equation 1 is the transfer function of an aberration-free optical system with a slit aperture operating in the Fraunhofer receiving plane. The aberration function thus comprises only a stochastic part, taken to be from a real, homogeneous Gaussian random process having zero mean, and covariance given by

$$\begin{aligned} R_W(\alpha_1 - \alpha_2) &= E\left[W\left(p + \frac{1}{2}\alpha_1\right)W\left(p - \frac{1}{2}\alpha_2\right)\right] \\ &= \sigma^2 r(|\alpha_1 - \alpha_2|) \end{aligned} \tag{2}$$

where $0 \leq |r| \leq 1$. Here and throughout, E[ ] is the expectation operator with respect to the ensemble. If we take the ensemble average of Eq. 1, we derive the well-known result[1,3]

$$E[T(\alpha)] = \exp\{-k^2\sigma^2[1-r(\alpha)]\}\left(1 - \frac{|\alpha|}{2}\right), \qquad |\alpha| \le 2$$

$$= 0 \qquad\qquad |\alpha| > 2 . \qquad (3)$$

Thus, the first moment behavior of $T(\alpha)$ factors into a deterministic part

$$T_D(\alpha) = 1 - \frac{|\alpha|}{2} \qquad (4)$$

and a stochastic component,

$$T_r(\alpha) = \exp\{-k^2\sigma^2[1-r(\alpha)]\} . \qquad (5)$$

Barakat[1] has shown that the factorization of $E[T(\alpha)]$ into stochastic and deterministic parts depends on the homogeneity of the wavefront errors, and occurs for the homogeneous case because, after commuting the operations of expectation and integration, the integrand does not depend on the integration variable and can be taken outside the integral. It is further shown in Ref. 1 that higher order averaged products of $T(\alpha)$ do not factor into stochastic and deterministic components. Since the treatment presented is not for merely $T(\alpha)$ but for $T(\alpha,\beta)$, i.e., the two-dimensional aperture case, we might expect a similar result when one spatial and one temporal dimension are included. In fact, the basic result does not depend on the number of dimensions over which the transfer function integral is computed, again provided that the covariance function of the Gaussian random

process representing the wavefront statistics is only a function of coordinate difference.

We now consider the time-dependent transfer function. Let $T_\tau(\alpha)$ be defined as

$$T_\tau(\alpha) = \frac{1}{2\tau} \int_0^\tau \int_{-\left(1 - \frac{|\alpha|}{2}\right)}^{1 - \frac{|\alpha|}{2}} \exp\left\{ki\left[W\left(p + \frac{\alpha}{2}, t\right) - W\left(p - \frac{\alpha}{2}, t\right)\right]\right\} dpdt \ . \tag{6}$$

To compute $E[T_\tau(\alpha)]$, we must first define a new, bivariate wavefront covariance function. Retaining our prior notation,

$$R_W(|\alpha_1-\alpha_2|, |t_1-t_2|) = E\left[W\left(p + \frac{1}{2}\alpha_1, t_1\right) W\left(p - \frac{1}{2}\alpha_2, t_2\right)\right]$$
$$= \sigma^2 r(|\alpha_1-\alpha_2|, |t_1-t_2|) \ . \tag{7}$$

Taking the expectation of Eq. 6, commuting expectation and integration, and evaluating, we have

$$E[T_\tau(\alpha)] = \frac{1}{2\tau} \int_0^\tau \int_{-\left(1 - \frac{|\alpha|}{2}\right)}^{1 - \frac{|\alpha|}{2}} \exp\{-k^2\sigma^2[1-r(\alpha,0)]\} dpdt$$
$$= \exp\{-k^2\sigma^2[1-r(\alpha,0)]\} \left(1 - \frac{|\alpha|}{2}\right), \quad |\alpha| \le 2 \ . \tag{8}$$

The general, formal extension for two spatial dimensions can be performed by inspection:

$$E[T_\tau(\alpha,\beta)] = \exp\{-k^2\sigma^2[1-r(\alpha,\beta,0)]\} \cdot T_D(\alpha,\beta) \tag{9}$$

with T_D the deterministic transfer function. Thus, we have the remarkably simple result that for the stationary zero mean Gaussian random wavefront considered, the expected value of the transfer function *does not depend* on the collection (integration) time.

Since it is well-known that short time exposure transfer functions are quite different from transfer functions involving long integration times, the observed difference must depend on the higher moments of $T_\tau(\alpha)$.

The second order averaged products of the transfer function at one frequency ($|\alpha_1| = |\alpha_2|$) are of two forms

$$E[|T_\tau(\alpha)|^2] = E\left[\left|\frac{1}{2\tau}\int_0^\tau \int_{-\left(1-\frac{|\alpha|}{2}\right)}^{1-\frac{|\alpha|}{2}} \exp\left\{ki\left[W\left(p+\frac{\alpha}{2}, t\right) - W\left(p-\frac{\alpha}{2}, t\right)\right]\right\} dp\,dt\right|^2\right] \tag{10}$$

and

$$E[T_\tau^2(\alpha)] = E\left[\left(\frac{1}{2\tau}\int_0^\tau \int_{-\left(1-\frac{|\alpha|}{2}\right)}^{1-\frac{|\alpha|}{2}} \exp\left\{ki\left[W\left(p+\frac{\alpha}{2},t\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left. - W\left(p-\frac{\alpha}{2},\ t\right)\right]\right\}dpdt\right)^2\right] . \tag{11}$$

These expressions are evaluated by converting the product of two-fold integrals into a four-fold integral of products, commuting the expectation and integration operations, and using the expression for the characteristic function of sums of normally distributed random variables[4]. After performing these operations we have

$$E[|T_\tau(\alpha)|^2] = \frac{1}{4\tau^2}\exp\{-2k^2\sigma^2[1-r(\alpha,0)]\}\int_0^\tau\int_0^\tau \iint_{-\left(1-\frac{|\alpha|}{2}\right)}^{1-\frac{|\alpha|}{2}}$$

$$\times \exp\{+k^2\sigma^2[2r(p_1-p_2,t_1-t_2) - r(p_1-p_2+\alpha,t_1-t_2)$$

$$- r(p_1-p_2-\alpha,t_1-t_2)]\}dp_1dp_2dt_1dt_2 \tag{12}$$

$$E[T_\tau^2(\alpha)] = \frac{1}{4\tau^2}\exp\{-2k^2\sigma^2[1-r(\alpha,0)]\}\int_0^\tau\int_0^\tau \iint_{-\left(1-\frac{|\alpha|}{2}\right)}^{1-\frac{|\alpha|}{2}}$$

$$\times \exp\{-k^2\sigma^2[2r(p_1-p_2,t_1-t_2) - r(p_1-p_2+\alpha,t_1-t_2)$$

$$- r(p_1-p_2-\alpha,t_1-t_2)]\}dp_1dp_2dt_1dt_2 . \tag{13}$$

As in the time-independent case, the two expressions are identical except for the sign change in the exponential of the integrand. Equations 12 and 13 can be used to compute the variances of the real and imaginary parts of the transfer function; the covariance between real and imaginary transfer function components is seen to vanish because Eq. 13 is purely real. Thus, dropping explicit dependence on α and τ for convenience, and using T_R and T_I to denote respectively the real and imaginary part of T,

$$E[|T|^2] = E[T_R^2] + E[T_I^2] \tag{14}$$

$$E[T^2] = E[T_R^2] - E[T_I^2] \tag{15}$$

$$\mathrm{var}\ T_R = \frac{1}{2} E[|T|^2] + \frac{1}{2} E[T^2] - (E[T])^2 \tag{16}$$

$$\mathrm{var}\ T_I = \frac{1}{2} E[|T|^2] - \frac{1}{2} E[T^2] \ . \tag{17}$$

For higher order averaged products, we again note the formal equivalence of the integrands of the transfer function $T_\tau(\alpha)$ and the transfer function $T(\alpha,\beta)$ for β fixed at zero:

$$T_\tau(\alpha) = \frac{1}{2\tau}\int_0^\tau \int_{-\left(1 - \frac{|\alpha|}{2}\right)}^{1 - \frac{|\alpha|}{2}} \exp\left\{ki\left[W\left(p + \frac{\alpha}{2}\ ,\ t\right) - W\left(p - \frac{\alpha}{2}\ ,\ t\right)\right]\right\}dpdt$$

$$T(\alpha,0) = \iint_{\text{Aperture}} \exp\left\{ki\left[W\left(p + \frac{\alpha}{2}\ ,\ q\right) - W\left(p - \frac{\alpha}{2}\ ,\ q\right)\right]\right\}dpdq \ . \tag{18}$$

Thus, the general expression for averaged products of the transfer function[1] derived for two spatial coordinates, is also that for the case of one spatial and one temporal coordinate, apart from changes in the limits of integration, and in normalization.

3. AMPLITUDE EFFECTS

The problem of the effect of random amplitude errors on the transfer function has historically been simplified by the assumption that the logarithmic amplitude fluctuations are normally distributed. Using this assumption, in addition to those of stationarity, and independence between the amplitude and phase fluctuations, it has been shown[5,6] that the expected value of the random transfer function component itself factors into one part due to phase errors and a second part due to amplitude errors. In the analysis below, we also consider the phase errors and log amplitude errors to be from zero mean, stationary Gaussian random processes, but we *do not* require that these processes be independent.

Let

$$T(\alpha) = \frac{1}{2}\int_{-\left(1 - \frac{|\alpha|}{2}\right)}^{1 - \frac{|\alpha|}{2}} A_0\left(p + \frac{\alpha}{2}\right) A_0\left(p - \frac{\alpha}{2}\right) \exp\left\{ki\left[W\left(p + \frac{\alpha}{2}\right) - W\left(p - \frac{\alpha}{2}\right)\right]\right\}dp \ . \qquad (19)$$

Setting $\psi = \ln\,(A_0/A)$, A chosen to normalize $T(\alpha)$, we have

$$T(\alpha) = \frac{1}{2} \int_{-\left(1 - \frac{|\alpha|}{2}\right)}^{1 - \frac{|\alpha|}{2}} \exp\left\{ki\left[W\left(p + \frac{\alpha}{2}\right) - W\left(p - \frac{\alpha}{2}\right)\right]\right.$$

$$\left. + \psi\left(p + \frac{\alpha}{2}\right) + \psi\left(p - \frac{\alpha}{2}\right)\right\} dp \; . \tag{20}$$

To compute $E[T(\alpha)]$, we may use the general characteristic function formula, evaluated for $n = 4$ terms. Somewhat simpler bookkeeping arises if we combine terms and use the formula for $n = 2$ terms. With

$$z_1 = kW\left(p + \frac{\alpha}{2}\right) - i\psi\left(p + \frac{\alpha}{2}\right)$$

$$z_2 = kW\left(p - \frac{\alpha}{2}\right) + i\psi\left(p - \frac{\alpha}{2}\right)$$

$$a_1 = 1 \;, \quad a_2 = -1 \;; \quad B_1 = p + \frac{\alpha}{2} \;, \quad B_2 = p - \frac{\alpha}{2}$$

and

$$E[\exp(ia_1 z_1 + ia_2 z_2)] = \exp\left\{-\frac{1}{2} \sum_{\ell=1}^{2} \sum_{m=1}^{2} a_\ell a_m \; E[Z_\ell(B_\ell) Z_m(B_m)]\right\} \tag{21}$$

there are four ($= n^2$) terms in the sum. These are

$\ell = m = 1$

$$k^2 E[W^2(p + \frac{\alpha}{2})] - E[\psi^2(p + \frac{\alpha}{2})] - 2iE[kW(p + \frac{\alpha}{2}) \; \psi(p + \frac{\alpha}{2})] \tag{22a}$$

$\ell = m = 2$

$$k^2E[W^2(p - \tfrac{\alpha}{2})] - E[\psi^2(p - \tfrac{\alpha}{2})] + 2iE[kW(p - \tfrac{\alpha}{2})\ \psi(p - \tfrac{\alpha}{2})] \tag{22b}$$

and, for $\ell = 1$, $m = 2$; $\ell = 2$, $m = 1$

$$-\{k^2E[W(p - \tfrac{\alpha}{2})\ W(p + \tfrac{\alpha}{2})] + E[\psi(p - \tfrac{\alpha}{2})\ \psi(p + \tfrac{\alpha}{2})]$$

$$- ikE[W(p - \tfrac{\alpha}{2})\ \psi(p + \tfrac{\alpha}{2})]$$

$$+ ikE[W(p + \tfrac{\alpha}{2})\ \psi(p - \tfrac{\alpha}{2})]\} \ . \tag{22c}$$

Simplifying,

$\ell = m = 1$

$$k^2\sigma_W^2 - \sigma_\psi^2 - 2ik\sigma_W\sigma_\psi r_{W\psi}(0) \tag{23a}$$

$\ell = m = 2$

$$k^2\sigma_W^2 - \sigma_\psi^2 + 2ik\sigma_W\sigma_\psi r_{W\psi}(0) \tag{23b}$$

$\ell = 1$, $m = 2$; $\ell = 2$; $m = 1$

$$-\{k^2\sigma_W^2 r_W(\alpha) + \sigma_\psi^2 r_\psi(\alpha) + ki\sigma_W\sigma_\psi[r_{W\psi}(\alpha) - r_{W\psi}(-\alpha)]\}. \tag{23c}$$

Here, the variances of the phase and log amplitude errors are given respectively by σ_W^2 and σ_ψ^2. The two autocorrelation functions and the cross correlation function are also subscripted with their pertinent variables. Combining terms as dictated by Eq. 21 and

performing the integration indicated by Eq. 20, which again does not depend functionally on p, we have

$$E[T(\alpha)] = \exp\{-k^2\sigma_w^2[1-r_w(\alpha)] + \sigma_\psi^2[1+r_\psi(\alpha)] - ki\sigma_w\sigma_r[r_{w\psi}(\alpha) - r_{\psi w}(-\alpha)]\} \times \left(1 - \frac{|\alpha|}{2}\right). \tag{24}$$

Equation 24 is just the expression usually derived under the assumption of independence between w and ψ, with the exception of the term involving the cross correlation function $r_{w\psi}(\alpha)$. Clearly, if $r_{w\psi}(\alpha)$ is zero for all values of α, this term vanishes. But if $r_{w\psi}$ is identically zero throughout, then w and ψ are uncorrelated and, by virtue of their being from stationary zero mean Gaussian random processes, they are also independent. However, we note it is *sufficient* but not *necessary* for $r_{w\psi}(\alpha)$ to be zero for the cross term to vanish; we require only that $r_{w\psi}(\alpha) = r_{w\psi}(-\alpha)$ for all α. Thus, within the stationary zero mean Gaussian assumption, the expected value of $T(\alpha)$ factors into a product of four components: one standard deterministic part, one part due to phase errors acting alone, one part due to amplitude errors acting alone, and a cross term which is of unit modulus and which depends on the odd part of $r_{w\psi}(\alpha)$ for its existence.

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